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GROUP RINGS OF GRADED RINGS. APPLICATIONS

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Introduction

One of the main problems in Graded Ring Theory is to see whether a graded module having a certain property has a similar property when regarded without grading. This problem has been taken into account in [9], using Internal and External Homogenization. The main drawback of both methods is that they apply mainly to the $\mathbb{Z}$-graded case. The purpose of this paper is to introduce a new technique for studying graded rings of type G, where G is an arbitrary group. This method will allow us to obtain several results concerning the above mentioned general problems.

In Section 1 we recall a series of notations and results of Graded Ring Theory.

In Section 2 we introduce the group ring of a graded ring of type G. More exactly, if $R=\bigoplus_{\sigma\in G} R_\sigma$ is a graded ring (with identity element) of type G, where G is an arbitrary group, we can define on the free R-module $R[G]=\{\sum_{g\in G}\lambda_g g \mid \lambda_g\in R\}$, with the basis $\{g \mid g\in G\}$, a new multiplication and a natural grading, which turn $R[G]$ into a strongly graded ring and R into a graded subring of $R[G]$. We note that the multiplication here introduced is different from the usual multiplication on group rings, when R is not graded. (However, the two operations coincide when G is an abelian group.) The idea which leads to introducing the graded ring $R[G]$ was suggested by the operation of External Homogenization for graded rings of type $\mathbb{Z}$: if $R=\bigoplus_{i\in Z} R_i$, then the polynomials ring $R[T]$ is a graded ring by $R[T]_n=\sum_{i+j=n} R_i T^j$ (see [9]). We introduce the graded $R[G]$-module $M[G]$, starting from a left graded R-module M. Proposition 2.1 and 2.2 give the basic properties of the graded ring $R[G]$ and the graded module $M[G]$.

In Section 3, using the graded ring $R[G]$ we prove the following result (see Theorem 3.1): if $M=\bigoplus_{\sigma\in G} M_\sigma$ is a graded left R-module which is gr-noetherian, and G is a strong polycyclic-by-finite group, then M is a noetherian R-module. This result extends the similar result given in [10] for the case G is finitely generated and abelian. The question remains open for the case when G is a polycyclic-by-finite group. This section ends with a result on the Krull dimension of the graded module $M=\bigoplus_{\sigma\in G} M_\sigma$ (see Theorem 3.2).

In Section 4 we give a graded version of Maschke's Theorem which allows us to prove a series of results concerning graded rings of type G, where G is a finite group (see Theorems 4.3, 4.5, 4.7, 4.10, 4.11, 4.13). These results generalize the similar results given in [8] for the case of strongly graded rings.

In Section 5 we deal with the study of the Jacobson radical of a graded R-module $M=\bigoplus_{\sigma\in G} M_\sigma$ (G is finite) using the graded Jacobson radical $J_g(M)$. The main result is contained in Theorem 5.4, which contains the following assertions: $J_g(M)\subseteq J(M)$, $nJ(M)\subseteq J_g(M)$ and if $x=\sum_{g\in G} x_g \in J(M)$ where $x_\sigma\in M_\sigma$, then $nx_g\in J(M)$, $(\forall) g\in G$, ($n=\operatorname{ord}(G)$). These assertions, in the particular case ${}_RM={}_RR$, constitute two conjectures posed by Bergman in his paper [2]. Another proof for the assertion $J_g(M)\subseteq J(M)$ is to be found in Corollary 4.14, using the study of gr-superfluous submodules in a graded module (see Theorem 4.13). The conjecture $J_g(R)\subseteq J(R)$ was proved for the first time (with different methods) by M. Cohen and S. Montgomery in their paper [3].

We remark that Theorems 4.5 and 5.4 allow us to provide a new simple proof of the following known results (see [2], [11], [12]): If $R=\bigoplus_{i\in\mathbb{Z}} R_i$ is a graded ring of type $\mathbb{Z}$ and $M=\bigoplus_{i\in\mathbb{Z}} M_i$ is a graded module, then the Jacobson radical $J(M)$ and the socle $s(M)$ are both graded submodules of M. (The fact that $J(R)$ is graded was proved by Bergman in his paper [2].)

In Section 6, the singular radical and prime radical are studied using the graded version of Maschke's Theorem (Theorem 6.7). Assertions (1) and (4) of Theorems 6.7 were proved for the first time, using different methods, by M. Cohen and S. Montgomery in [3]. Corollaries 6.4 and 6.6 complete the results of this paragraph.

1. Notations and preliminaries

All rings considered in this paper will be unitary. If R is a ring, by an R-module we will mean a left R-module, and we will denote the category of R-modules by R-mod. If G is a group and $R=\bigoplus_{\sigma\in G} R_\sigma$ is a graded ring of type G, the category of graded R-modules will be denoted by R-gr. It is well known that R-gr is a Grothendieck category [9].

If $M=\bigoplus_{\sigma\in G} M_\sigma$ is a graded R-module, we will let $\underline{M}$ denote the underlying R-module of M and by $h(M)$ we will mean the set of all homogeneous elements of M, i.e. $h(M)=\bigcup_{\sigma\in G} M_\sigma$. For any submodule of M we define $(N)_g=\bigoplus_{\sigma\in G}(N\cap M_\sigma)$; we say that N is a graded submodule if $N=(N)_g$, more generally $(N)_g$ is the largest graded submodule of M contained in N.

If $M=\bigoplus_{\sigma\in G} M_\sigma$, $N=\bigoplus_{\sigma\in G} N_\sigma$ are two graded R-modules, $\operatorname{Hom}_{R\text{-gr}}(M,N)$ is the set of morphisms in the category R-gr from M to N, i.e.

$$\operatorname{Hom}_{R\text{-gr}}(M,N)=\{f: M\to N \mid f \text{ is } R\text{-linear and } f(M_\sigma)\subseteq N_\sigma, \forall\sigma\in G\}.$$

If $M=\bigoplus_{\lambda\in G}M_\lambda$ is a graded R-module and $\sigma\in G$, then $M(\sigma)$ is the graded module obtained from M by putting $M(\sigma)_\lambda=M_{\lambda\sigma}$; the graded module $M(\sigma)$ is called the *σ-suspension* of M [9]. It is well known [9] that the mapping $M\to M(\sigma)$ defines a functor from R-gr to R-gr which is an equivalence of categories.

If H is a subgroup of G and $R=\bigoplus_{\sigma\in G}R_\sigma$ is a graded ring of type G, $R^{(H)}=\bigoplus_{h\in G}R_h$ is a graded ring of type H.

Moreover, if $M=\bigoplus_{\sigma\in G}M_\sigma$ is a graded R-module and $(\sigma_i)_{i\in I}$ is a set of representatives for the right H-cosets of G, then for each σ_i we put $M^{(H\sigma_i)}=\bigoplus_{h\in H}M_{h\sigma_i}$. Clearly $M^{(H\sigma_i)}$ is a graded $R^{(H)}$-module and $M=\bigoplus_{i\in I}M^{(H\sigma_i)}$ [9].

If $H\lhd G$ is a normal subgroup of G, the G-grading on R induces a G/H-grading on $R: R=\bigoplus_{\hat\sigma\in G/H}R_{\hat\sigma}$ where $R_{\hat\sigma}=\bigoplus_{h\in H}R_{h\sigma}$ (see [9]).

If $R=\bigoplus_{\sigma\in G}R_\sigma$ is a graded ring, we say that R is a strongly graded ring if $R_\sigma R_\tau=R_{\sigma\tau}$ for any $\sigma,\tau\in G$. It is well known [9] that R is a strongly graded ring if and only if $R_\sigma R_{\sigma^{-1}}=R_e$ for any $\sigma\in G$ (e is the identity of the group G).

If $H<G$ is a subgroup of G, hen $R^{(H)}$ is also a strongly graded ring.

The connection between the categories R-gr and R_e-mod is given by the following:

Theorem P ([5, Theorem 2.8] or [9, Theorem I.3.4]). *Let $R=\bigoplus_{\sigma\in G}R_\sigma$ be a strongly graded ring. Then the functor $R\otimes_{R_e}\cdot : R_e\text{-mod}\to R\text{-gr}$ given by $M\to R\otimes_{R_e}M$ where $M\in R_e\text{-mod}$ and $R\otimes_{R_e}M$ is a graded R-module by the grading $(R\otimes_{R_e}M)_\sigma=R_\sigma\otimes_{R_e}M$, is an equivalence. Its inverse is the functor $(\cdot)_e: R\text{-gr}\to R_e\text{-mod}$ given by $M\to M_e$ where $M\in R\text{-gr}$ and $M=\bigoplus_{\sigma\in G}M_\sigma$.*

2. Group ring of graded rings

If $R=\bigoplus_{\sigma\in G}R_\sigma$ is a graded ring of type G we denote by $R[G]$ the left free R-module with the basis $\{\sigma\mid\sigma\in G\}$, i.e. $R[G]=\{\sum_{g\in G}\lambda_g g\mid\lambda_g\in R\}$.

We define for every $\sigma\in G$:

$$(R[G])_\sigma=\sum_{\lambda\mu=\sigma}R_\lambda\mu=\sum_{\tau\in G}R_{\sigma\tau^{-1}}\tau=\bigoplus_{\tau\in G}R_{\sigma\tau^{-1}}\tau.$$

Clearly $R[G]=\bigoplus_{\sigma\in G}(R[G])_\sigma$ and $R[G]$ is a graded R-module with the grading $\{(R[G])_\sigma\}_{\sigma\in G}$.

For the elements $\lambda_\sigma\tau$ and $\lambda_{\sigma'}\tau'$ where $\lambda_\sigma\in R_\sigma$, $\lambda_{\sigma'}\in R_{\sigma'}$, we define their product by

$$(*)\qquad (\lambda_\sigma\tau)\cdot(\lambda_{\sigma'}\tau')=\lambda_\sigma\lambda_{\sigma'}(\sigma'^{-1}\tau\sigma'\tau').$$

Since every element of $R[G]$ is an unique sum of elements of the form $\lambda_\sigma\tau$ with $\lambda_\sigma\in R_\sigma$, the product (*) may be extended to a multiplication on $R[G]$.

Proposition 2.1. *With the above notations, we have the following assertions:*

(1) *The multiplication defined by* (*) *is associative.*

(2) $R[G]$ *is a strongly graded ring with the grading* $\{(R[G])_\sigma, \sigma \in G\}$.

(3) $R[G]$ *is also a free right R-module with basis* $\{\sigma \mid \sigma \in G\}$.

(4) $(R[G])_e = \sum_{\sigma\in G} R_{\sigma^{-1}}\sigma$ *and* $\varphi: R \to (R[G])_e$, $\varphi(\sum_{\sigma\in G}\lambda_\sigma) = \sum_{\sigma\in G}\lambda_\sigma\sigma^{-1}$, *where* $\lambda_\sigma \in R$, *is a ring isomorphism.*

(5) *If* $H \lhd G$ *is a normal subgroup of G, then* $R[H]$ *is a graded subring of* $R[G]$. (*Here* $R[H] = \{\sum_{i=1}^n \lambda_i h_i \mid \lambda_i \in R, h_i \in H\}$.

(6) *If I is a graded left ideal of R, then* $I[G]$ *is a left graded ideal of* $R[G]$ *and* $I[G] \cap (R[G])_e = \varphi(I)$.

Proof. We consider the elements $\{\lambda_{\sigma_i}\tau_i\}_{i=1,2,3}$ where $\lambda_{\sigma_i} \in R_{\sigma_i}$. Then we have

$$\begin{aligned}(\lambda_{\sigma_1}\tau_1)(\lambda_{\sigma_2}\tau_2)(\lambda_{\sigma_3}\tau_3) &= (\lambda_{\sigma_1}\tau_1)(\lambda_{\sigma_2}\lambda_{\sigma_3}\sigma_3^{-1}\tau_2\sigma_3\tau_3)\\ &= \lambda_{\sigma_1}\lambda_{\sigma_2}\lambda_{\sigma_3}(\sigma_2\sigma_3)^{-1}\tau_1(\sigma_2\sigma_3)\sigma_3^{-1}\tau_2\sigma_3\tau_3\\ &= (\lambda_{\sigma_1}\lambda_{\sigma_2}\lambda_{\sigma_3})(\sigma_3^{-1}\sigma_2^{-1}\tau_1\sigma_2\tau_2\sigma_3\tau_3).\end{aligned}$$

On the other hand

$$\begin{aligned}(\lambda_{\sigma_1}\tau_1)(\lambda_{\sigma_2}\tau_2)(\lambda_{\sigma_3}\tau_3) &= (\lambda_{\sigma_1}\lambda_{\sigma_2}\sigma_2^{-1}\tau_1\sigma_2\tau_2)(\lambda_{\sigma_3}\tau_3)\\ &= (\lambda_{\sigma_1}\lambda_{\sigma_2}\lambda_{y_3})\tau_3^{-1}(\sigma_2^{-1}\tau_1\sigma_2\tau_2)\sigma_3\tau_3\\ &= (\lambda_{\sigma_1}\lambda_{\sigma_2}\lambda_{\sigma_3})(\sigma_3^{-1}\sigma_2^{-1}\tau_1\sigma_2\tau_2\sigma_3\tau_3).\end{aligned}$$

Hence

$$(\lambda_{\sigma_1}\tau_1)[(\lambda_{\sigma_2}\tau_2)(\lambda_{\sigma_3}\tau_3)] = [(\lambda_{\sigma_1}\tau_1)(\lambda_{\sigma_2}\tau_2)](\lambda_{\sigma_3}\tau_3)$$

and so the multiplication of $R[G]$ is associative.

(2) In order to prove that $(R[G])_\sigma(R[G])_{\sigma'} \subseteq (R[G])_{\sigma\sigma'}$ it is enough to show that $(R_{\sigma\tau^{-1}}\tau)(R_{\sigma'\tau'^{-1}}\tau') \subseteq (R[G])_{\sigma\sigma'}$. Indeed, if $\lambda_{\sigma\tau^{-1}} \in R_{\sigma\tau^{-1}}$ and $\lambda_{\sigma'\tau'^{-1}} \in R_{\sigma'\tau'^{-1}}$ we have

$$\begin{aligned}(\lambda_{\sigma\tau^{-1}}\tau)(\lambda_{\sigma'\tau'^{-1}}\tau') &= (\lambda_{\sigma\tau^{-1}}\lambda_{\sigma'\tau'^{-1}})(\tau'\sigma'^{-1}\tau\sigma'\tau'^{-1}\tau')\\ &= (\lambda_{\sigma\tau^{-1}}\lambda_{\sigma'\sigma'^{-1}})(\tau'\sigma'^{-1}\tau\sigma') \in R_{\sigma\tau^{-1}\sigma'\tau'^{-1}}\tau'\sigma'^{-1}\tau\sigma'\\ &= R_{\sigma\sigma'(\tau'\sigma'^{-1}\tau\sigma')^{-1}}(\tau'\sigma'^{-1}\tau\sigma') \subseteq (R[G])_{\sigma\sigma'}.\end{aligned}$$

Since $R_e\sigma \subseteq (R[G])_\sigma$, $1\cdot\sigma \in (R[G])_\sigma$. Analogously $1\cdot\sigma^{-1} \in (R[G])_{\sigma^{-1}}$ and therefore $(R[G])_\sigma \cdot (R[G])_{\sigma^{-1}} = (R[G])_e$. Hence $R[G]$ is a strongly graded ring. (In fact $R[G]$ is a crossed product [9].)

(3) We consider the sum $\sum_{i=1}^n g_i\lambda_i = 0$ where $\lambda_i \in R$. Since $\{g_i\}_{i=1,\dots,n}$ are homogeneous elements, we may suppose the λ_i are homogeneous elements of R. Assume that $\lambda_i = \lambda_{\sigma_i} \in R_{\sigma_i}$. But $\sum g_i\lambda_{\sigma_i} = \sum \lambda_{\sigma_i}(\sigma_i^{-1}g_i\sigma_i) = 0$.

Since $\deg(g_1\lambda_{\sigma_1}) = \deg(g_2\lambda_{\sigma_2}) = \cdots = \deg(g_n\lambda_{\sigma_n})$, $g_1\sigma_1 = g_2\sigma_2 = \cdots = g_n\sigma_n$ and therefore the elements $\{\sigma_i^{-1}g_i\sigma_i\}_{i=1,\dots,n}$ are pairwise distinct. Then from the equality $\sum \lambda_{\sigma_i}(\sigma_i^{-1}g_i\sigma_i) = 0$ we obtain $\lambda_{\sigma_i} = 0$, $i = 1,\dots,n$.

(4) If $\lambda_\sigma \in R_\sigma$, $\lambda_\tau \in R_\tau$, we have $\varphi(\lambda_\sigma\lambda_\tau) = \lambda_\sigma\lambda_\tau(\sigma\tau)^{-1} = (\lambda_\sigma\sigma^{-1})(\lambda_\tau\tau^{-1}) = \varphi(\lambda_\sigma)\cdot\varphi(\lambda_\tau)$ and therefore φ is a ring homomorphism. It is clear that φ is an isomorphism.

(5) Let $\lambda_g h, \lambda_{g'} h' \in R[H]$ where $\lambda_g \in R_g$, $\lambda_{g'} \in R_{g'}$ and $h, h' \in H$. Then $(\lambda_g h)(\lambda_{g'} h') = \lambda_g \lambda_{g'} (g'^{-1} h g' h') \in R[H]$. Hence $R[H]$ is a subring of $R[G]$. It is clear that $R[H] = \bigoplus_{g \in G}(R[H] \cap (R[G])_g)$ which shows that $R[H]$ is a graded subring of $R[G]$.

$$R[H]_g = \left\{ \sum_{i=1}^{n} \lambda_i h_i \,\middle|\, \lambda_i \in R_{g_i} \text{ with } g_i h_i = g \ (1 \le i \le n) \right\}$$

(6) The inclusion $\varphi(I) \subseteq I[G] \cap (R[G])$ is clear. Now, let $x \in I[G] \cap (R[G])_e$. Then $x = \sum_{i=1}^{n} \lambda_i g_i$, $\lambda_i \in I$. Since $x \in \sum_{g \in G} R_{g^{-1}} g$, then $\lambda_i \in R_{g_i^{-1}}$, and it follows that $x = \varphi(\sum_{i=1}^{n} \lambda_i) \in \varphi(I)$.

Remarks. (1) One can also consider the usual multiplication on $R[G]: (a_g g)(b_h h) = a_g b_h (gh)$, where $a_g, b_h \in R$, but with this multiplication $R[G]$ is not a graded ring with the above grading.

(2) It is now easy to see that $\forall g \in G$ commute with any element of $R[G]_e$ and therefore $R[G]$ is the group ring of $(R[G])_e$, by the group G in the classical sense.

Let now $M = \bigoplus_{\sigma \in G} M_\sigma$ be a left graded R-module. We denote $M[G] = \bigoplus_{g \in G} M_g$ where $M_g = M$, $\forall g \in G$ which is a left R-module. We can identify

$$M[G] = \left\{ \sum_{g \in G} m_g g \,\middle|\, m_g \in M \right\}.$$

The family of abelian groups $(M[G])_\sigma = \sum_{\lambda\mu = \sigma} M_\lambda \mu$ defines on $M[G]$ a graduation as an R-module. We define on $M[G]$ the multiplication by the rule:

$$(a_\sigma \tau)(m_g h) = (a_\sigma m_g)(g^{-1} \tau g h), \qquad a_\sigma \in R_\sigma, \quad m_g \in M_g.$$

Proposition 2.2. *With the above notations we have:*

(1) *$M[G]$ is a graded $R[G]$-module with the graduation $\{(M[G])_\sigma \mid \sigma \in G\}$.*

(2) *$(M[G])_e = \sum_{\sigma \in G} M_{\sigma^{-1}} \sigma$ and the mapping $\psi : M \to (M[G])_e$ where $\psi(\sum_{g \in G} m_g) = \sum m_g g^{-1}$, $m_g \in M_g$ is a φ-isomorphism.*

(3) *If $H \lhd G$ is a normal subgroup of G, then $M[H]$ is a graded $R[H]$-module.*

(4) *If $N \subseteq M$ is a graded submodule of M, then $N[G]$ is a graded submodule of $M[G]$ and $N[G] \cap (M[G])_e = \psi(N)$.*

(5) *$M[G]$ is isomorphic in the category R-gr to $\bigoplus_{g \in G} M(g^{-1})$.*

(6) *If $N \subset M$ is an R-submodule of M (non-graded), then $R[G]\psi(N) \cap M = (N)_g$.*

Proof. The statements (1), (2), (3) and (4), are proved exactly as in Proposition 2.1.

(5) The mapping $\sum_{g \in G} m_g g \to (m_g)_{g \in G}$ is an R-isomorphism in R-gr from $M[G]$ to $\bigoplus_{g \in G} M(g^{-1})$.

(6) If $x_g \in N \cap M_g$, since $x_g = g(x_g g^{-1})$, then $x_g \in R[G]\psi(N)$ and therefore $(N)_g \subset M \cap R[G]\psi(N)$.

Conversely, let $x = \sum_{g \in G} x_g$ be an element of N. We denote by $x^* = \psi(x) = \sum_{g \in G} x_g g^{-1} \in \psi(N)$. If $u = \sum_{i=1}^{n} \lambda_{\sigma_i} \tau_i$, $\lambda_{\sigma_i} \in R_{\sigma_i}$, is homogeneous of degree θ, then $\sigma_1 \tau_1 = \sigma_2 \tau_2 = \cdots = \sigma_n \tau_n = \theta$. We have

$$ux^* = \sum_{i=1}^{n} \sum_{g\in G} \lambda_{\sigma_i}\tau_i x_g g^{-1}$$
$$= \sum_{i=1}^{n} \lambda_{\sigma_i} \sum_{g\in G} x_g g^{-1}\tau_i$$
$$= \sum_{i=1}^{n} \sum_{g\in G} (\lambda_{\sigma_i} x_g)(\sigma_i g)^{-1}\sigma_i\tau_i$$
$$= \sum_{i=1}^{n} \sum_{g\in G} (\lambda\sigma_i x_g)(\sigma_i g)^{-1}\theta.$$

We put $x_i = \sum_{g\in G} \lambda_{\sigma_i} x_g$ and $y = \sum_{i=1}^{n} x_i$. It may be easily seen that $x_i \in N$ and hence $y \in N$. Now it is straightforward to check that $ux^* = \theta \sum_{i=1}^{n} x_i^* = \theta y^*$.

If $\alpha \in R[G]\psi(N)$ is homogeneous of degree θ, then there exists $z_i \in N$ and $v_i \in R[G]$ with $\deg v_i = \theta$ such that $\alpha = \sum_{k=1}^{m} v_k z_k^*$. It follows that there exists $z \in N$ such that $\alpha = \theta z^*$. Now, if $\alpha \in R[G]\psi(N) \cap M$, it follows that $\alpha \in M_\theta$. If $z = \sum_{g\in G} z_g$, with $z_g \in M_g$, then $\alpha = \theta z^* = \theta \sum_{g\in G} z_g g^{-1} = \sum_{g\in G} z_g g^{-1}\theta$ and since $\alpha \in M_\theta$, we must have $z_g = 0$ for $g \neq \theta$ and hence $\alpha = z_\theta = z$ which shows that $\alpha \in (N)_g$.

3. Graded rings with finiteness conditions

If $R = \bigoplus_{\sigma\in G} R_\sigma$ is a graded ring of type G and $M = \bigoplus_{\sigma\in G} M_\sigma$ is a graded R-module, then M is said to be gr-G-noetherian if M satisfies the ascending chain condition on graded submodules. It is straightforward to check that M is gr-G-noetherian if and only if each graded submodule of M is finitely generated.

The group G is said to be a *strong polycyclic-by-finite group* if G has a finite series $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$ such that G_i is a normal subgroup of G for each $i = 0, 1, \ldots, n$ and the quotients G_{i+1}/G_i are either finite or cyclic. (If G has a finite subnormal series $\{e\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ such that G_{i+1}/G_i are either finite or cyclic, the group G is called polycyclic-by-finite (see [13]).)

The main result of this section is the following:

Theorem 3.1. *Suppose that G is a strong polycyclic-by-finite group and $M = \bigoplus_{\sigma\in G} M_\sigma$ is a graded R-module. Then the following assertions are equivalent:*

(1) *M is gr-G-noetherian.*

(2) *$\underline{M}$ is a noetherian R-module.*

The proof of this theorem requires some preliminary results:

Lemma 1 [9]. *Let $H \lhd G$ be a subgroup of G and let $\{\sigma_i\}_{i\in I}$ be a set of representatives for the right H-cosets of G. If M is Gr-noetherian, then $M^{(H\sigma_i)} = \bigoplus_{h\in H} M_{h\sigma_i}$ is gr-H-noetherian over the ring $R^{(H)}$.*

In particular, if $[G:H] < \infty$, then M is gr-H-noetherian over the ring $R^{(H)}$.

Proof. See [9, Corollar... 3].

Lemma 2. *Let $\langle\sigma\rangle \triangleleft G$ ... infinite cyclic normal subgroup of G. We denote by $H_\sigma = \{g \in G \mid$ there exists ... 0 such that $g^{-1}\sigma g = \sigma^k\}$. Then:*

(a) *If $g \in G$, $g^{-1}\sigma g = \sigma^{\pm 1}$.*

(b) *$H_\sigma = \{g \in G \mid g^{-1}\sigma g = \sigma\}$.*

(c) *$H_\sigma = H_{\sigma^{-1}}$.*

(d) *$H_\sigma \triangleleft G$ and $[G:H_\sigma] \le 2$.*

Proof. (a) Since $\langle\sigma\rangle$ is a normal subgroup, we have for each $g \in G$, $g^{-1}\sigma g = \sigma^t$ and $g\sigma g^{-1} = \sigma^u$, where $t, u \in Z$. Hence $g\sigma^t g^{-1} = \sigma$ or $(g\sigma g^{-1})^t = \sigma$. Hence $\sigma^{ut} = \sigma$ and therefore $ut = 1$. It follows that $t = \pm 1$.

(b) We apply (a).

(c) Since $g^{-1}\sigma g = \sigma \Leftrightarrow (g^{-1}\sigma g)^{-1} = \sigma^{-1} \Leftrightarrow g^{-1}\sigma^{-1}g = \sigma^{-1}$, it follows that $H_\sigma = H_{\sigma^{-1}}$.

(d) If $g, h \in H_\sigma$ it is clear that $gh \in H_\sigma$. Now if $g \in H_\sigma$, $g^{-1}\sigma g = \sigma$ or $g\sigma g^{-1} = \sigma$ or $(g^{-1})^{-1}\sigma^{-1} = \sigma$ and hence $g^{-1} \in H_\sigma$. Thus h_σ is a subgroup of G. Let $g \in G$, $h \in H_\sigma$; if $g \in H_\sigma$, then $g^{-1}hg \in H_\sigma$; if $g \notin H_\sigma$, then by assertion (a), $g\sigma g^{-1} = \sigma^{-1}$ and hence $g^{-1}\sigma^{-1}g = \sigma$. Then $(g^{-1}hg)^{-1}\sigma(g^{-1}hg) = g^{-1}h^{-1}(g\sigma g^{-1})hg = g^{-1}h^{-1}\sigma^{-1}hg = g^{-1}\sigma^{-1}g = \sigma$ and hence $g^{-1}hg \in H_\sigma$. Thus H_σ is a normal subgroup of G. Assume that $[G:H] \ge 3$. There exist $\hat{g}_1, \hat{g}_2 \in G/H_\sigma$ such that $\hat{g}_1 \ne \hat{g}_2$ and $\hat{g}_1 \ne e$, $\hat{g}_2 \ne e$. Thus $g_1, g_2 \notin H_\sigma$ and by assertion (a) we have $g_1^{-1}\sigma g_1 = \sigma^{-1}$ and $g_2^{-1}\sigma g_2 = \sigma^{-1}$. Then $(g_1g_2^{-1})^{-1}\sigma(g_1g_2^{-1})^{-1} = g_2(g_1^{-1}\sigma g_1)g_2^{-1} = g_2\sigma^{-1}g_2^{-1} = \sigma$ and therefore $g_1g_2^{-1} \in H_\sigma$. Hence $\hat{g}_1 = \hat{g}_2$, a contradiction.

Lemma 3. *Let $H \triangleleft K \triangleleft G$ be two normal subgroups of G such that $K/H = \langle\hat{\sigma}\rangle$ is an infinite cyclic group. Let $M = \bigoplus_{g \in G} M_g$ be a graded R-module such that $M[H]$ is a gr-noetherian module over the ring $R[H]$.*

Then $M[K]$ is a gr-noetherian module over the ring $R[K]$.

Proof. By Proposition 2.2, $M[H]$ is a graded $R[H]$-module of type G. If $y = \sum_{i=1}^n m_ih_i$, $m_i \in M$, $h_i \in G$, is an arbitrary element of $M[G]$ and $g \in G$, then by yg we understand the element $yg = \sum_{i=1}^n m_i(h_ig) \in M[G]$. With these notations we have

$$y(gg') = (yg)g' \quad \text{for any } g, g' \in G.$$

Now, since $K = \langle H, \sigma, \sigma^{-1}\rangle$,

$$M[K] = M[H][\sigma, \sigma^{-1}] = \left\{ \sum_{i=-m}^{n} x_i\sigma^i \,\middle|\, x_i \in M[H], m, n \ge 0 \right\}.$$

We denote

$$H_\sigma = \{g \in G \mid \hat{g}^{-1}\hat{\sigma}\hat{g} = \hat{\sigma} \text{ in } G/H\} = \{g \in G \mid g^{-1}\sigma g = hg, h \in H\}.$$

By Lemma 2 we have $K \triangleleft H_\sigma \triangleleft G$, $H_\sigma = H_{\sigma^{-1}}$ and $[G:H_\sigma] \le 2$.

We denote $S=R[H]^{(H_\sigma)}$ and $N=M[H]^{(H_\sigma)}$. It is clear that S is a graded ring of type H_σ and N is a graded S-module.

Sublemma 1. *Let $y=\sum_{i=1}^{n} m_{g_i}h_i\in M[H]^{(H_\sigma)}$ be a homogeneous element of degree τ, where $m_{g_i}\in M_{g_i}$, $h_i\in H$. Let $h\in H$ be such that $\tau^{-1}\sigma\tau=h\sigma$ and $h'=\tau h^{-1}\tau^{-1}$.*

Then $h'\in H$, $(h'\sigma)y=y\sigma$ and $\sigma y=y(h\sigma)=(yh)\sigma$.

Proof. By Lemma 2, there exists $h\in H$ such that $\tau^{-1}\sigma\tau=h\sigma$ and since H is a normal subgroup, $h'\in H$. Because $\deg(y)=\tau$, $g_1h_1=g_2h_2=\cdots=g_nh_n=\tau\in H_\sigma$. Thus

$$\begin{aligned}(h'\sigma)y&=\sum_{i=1}^{n}(h'\sigma)(m_{g_i}h_i)=\sum_{i=1}^{n}m_{g_i}(g_i^{-1}h'\sigma g_ih_i)\\&=\sum_{i=1}^{n}m_{g_i}(g_i^{-1}h'\sigma\tau)=\sum_{i=1}^{n}m_{g_i}(g_i^{-1}\tau h^{-1}\tau^{-1}\tau h\sigma)\\&=\sum_{i=1}^{n}m_{g_i}(g_i^{-1}\tau\sigma)=\sum_{i=1}^{n}m_{g_i}(g_i^{-1}\tau)\sigma=\sum_{i=1}^{n}m_{g_i}h_i\sigma=y\sigma.\end{aligned}$$

Analogously we show that $\sigma y=(yh)\sigma$.

Now, from Sublemma 1, $S[\sigma]=\{\sum_{i=0}^{m}s_i\sigma^i\mid s_i\in S, m\geq 0\}$ is a graded ring of $R[G]$ and $N[\sigma]$ is a graded module over the ring $S[\sigma]$.

Step I. $N[\sigma]$ is a gr-noetherian module over the ring $S[\sigma]$.

Indeed, let $X\subset N[\sigma]$ be a graded submodule of $N[\sigma]$. We denote, for each $n\geq 0$,

$$X_n=\{y\in N\mid \exists x\in X: x=y_0+y_1\sigma+\cdots+y_{n-1}\sigma^{n-1}+y\sigma^n, y_i\in N\}.$$

It is easy to see that X_n is an S-submodule of N. Because X is a graded submodule and σ is a homogeneous element, X_n is moreover a graded submodule of N.

Now, let $y\in X_n$ be a homogeneous element; there exists a homogeneous element $x\in X$ with $x=y_0+y_1\sigma+\cdots+y\sigma^n$, $y_i\in N$. By Sublemma 1, $(h'\sigma)x\in X$ and $(h'\sigma)x=(h'\sigma)y_0+(h'\sigma)y_1\sigma+\cdots+(h'\sigma)y\sigma^n=(h'\sigma)y_0+\cdots+y\sigma^{n+1}$ and therefore $y\in X_{n+1}$. It means that $X_0\subseteq X_1\subseteq\cdots\subseteq X_n\subseteq\cdots$ is an ascending chain of graded submodules of N. Since $M[H]$ is a gr-noetherian module, it follows by hypothesis, by Lemma 1, that N is a gr-noetherian S-module. Thus there exists $n\geq 0$ such that $X_n=X_{n+1}=\cdots$.

For $i=0,1,\dots,n$, let $\{y_{ij}\}_{j=1,\dots,k_i}$ be some finitely many homogeneous elements of N that generate X_i as S-module and choose $x_{ij}\in X$ with

$$x_{ij}=y_{ij}\sigma^i+\sum_{k=0}^{i-1}z_{kj}\sigma^k,\quad z_{kj}\in N.$$

Let $x\in X$ be a homogeneous element, $x=y_0+y_1\sigma+\cdots+y_p\sigma^p$, where $y_i\in N$ are homogeneous. By induction on the degrees of σ we show that x is a left $S[\sigma]$ linear sum of elements $\{x_{ij}\mid i=0,\dots,n; j=1,\dots,k\}$.

Indeed, $y_p\in X_p$. If $p\geq n$, then $y_p\in X_n$ so that $y_p=\sum_{j=1}^{k}a_{nj}y_{nj}$, $a_{nj}\in S$. By Sublemma 1, $y_p\sigma^p = y_p\sigma^{p-n}\sigma^n = \sum_j \lambda_{nj}(y_{nj}\sigma^n)$ where $\lambda_{nj} \in S$. Hence $y_p\sigma^p = \sum_j \lambda_{nj}x_{ij}$ + lower degree terms. So $x - \sum_j \lambda_{nj}x_{ij} \in X$ and $x - \sum \lambda_{nj}x_{ij} = y'_0 + y'_1\sigma + \cdots + y'_m\sigma^m$ where $m<n$ and $y'_i \in N$. Now apply the induction hypothesis.

Step II. $N[\sigma,\sigma^{-1}]$ is a gr-noetherian module over the ring $S[\sigma,\sigma^{-1}]$.

Indeed, let X be a graded submodule of $N[\sigma,\sigma^{-1}]$. Then $X\cap N[\sigma]$ is a graded $S[\sigma]$-submodule of $N[\sigma]$ and therefore it is generated by finitely many homogeneous elements $x_1,\dots,x_n \in X\cap N[\sigma]$. If $x\in X$ is a homogeneous element, then $x = \sum_{k=-m}^{l} n_k\sigma^k$, $n_k \in N$ are homogeneous elements. By Sublemma 1, $\sigma^m x \in N[\sigma]$ and therefore $\sigma^m x = \sum_{i=1}^{n} \lambda_i x_i$, $\lambda_i \in S[\sigma]$, so $x = \sum_{i=1}^{n} (\sigma^{-m}\lambda_i)x_i$ and hence X is finitely generated.

In addition, if $X \subseteq Y$ are two graded $S[\sigma,\sigma^{-1}]$-submodules of $N[\sigma,\sigma^{-1}]$, then we have $X = Y \Leftrightarrow X\cap N[\sigma] = Y\cap N[\sigma]$. We denote $P = M[H]^{(G-H_\sigma)}$. By Lemma 1 and the hypothesis, P is a gr-noetherian module over the ring $R[H]^{(H_\sigma)}$.

Step III. $P[\sigma,\sigma^{-1}]$ is a gr-noetherian module over the ring $S[\sigma,\sigma^{-1}]$.

Sublemma 2. *Let $y = \sum m_{g_i}h_i \in M[H]^{(G-H_\sigma)}$ be a homogeneous element of degree τ where $m_{g_i} \in M_{g_i}$, $h_i \in H$. Let $h \in H$ be such that $\tau^{-1}\sigma\tau = h\sigma^{-1}$ and $h' = \tau h^{-1}\tau^{-1}$.*

Then $h' \in H$, $(h'\sigma)y = y\sigma^{-1}$ and $\sigma y = y(h\sigma^{-1})$.

Proof. We have $g_1h_1 = g_2h_2 = \cdots = g_nh_n = \tau$ where $\tau \notin H_\sigma$. Since $H \subseteq H_\sigma$, $g_i \notin H_\sigma$. By Lemma 2, there exists h such that $\tau^{-1}\sigma\tau = h\sigma^{-1}$. Exactly as in Sublemma 1, we show the two equalities.

Now, by Sublemma 2, $P[\sigma^{-1}]$ is an $S[\sigma]$-graded module. Exactly as in Step I, we show that $P[\sigma^{-1}]$ is a gr-noetherian module over the ring S.

Now, proceeding as in Step II, we prove that $P[\sigma,\sigma^{-1}]$ is an $S[\sigma,\sigma^{-1}]$-gr-noetherian module.

Step IV. $M[K]$ is an $R[K]$-gr-noetherian module.

Indeed, $M[K] = N[\sigma,\sigma^{-1}] \oplus P[\sigma,\sigma^{-1}]$. By Steps II and III, $M[K]$ is a gr-noetherian module over the ring $R[K]$.

Proof of Theorem 3.1. The implication $(2)\Rightarrow(1)$ is clear.

$(1)\Rightarrow(2)$. Let $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$ be a normal series for G. By induction on $0 \le i \le n$ we show that $M[G_i]$ is a gr-noetherian module over the ring $R[G_i]$. If $i=0$, then $M[G_0] = M$ and therefore the statement is obvious. We suppose that $M[G_i]$ is a gr-noetherian module over the ring $R[G_i]$. If G_{i+1}/G_i is a finite group, we assume that $G_{i+1}/G_i = \{\hat\sigma_1,\dots,\hat\sigma_t\}$; then $M[G_{i+1}] = M[G_i]\sigma_1 + \cdots + M[G_i]\sigma_t$. It is clear that the $M[G_i]\sigma_i$ are $R[G_i]$-gr-noetherian modules, so $M[G_{i+1}]$ is $R[G_i]$-gr-noetherian. Hence $M[G_{i+1}]$ is an $R[G_{i+1}]$-gr-noetherian module.

If G_{i+1}/G_i is an infinite cyclic group, by Lemma 3, we obtain the statement.

Consequently, $M[G]$ is an $R[G]$-gr-noetherian module.

By Theorem P, it follows that $(M[G])_e$ is an $(R[G])_e$-noetherian module. Now apply Proposition 2.2 and obtain that M is an R-noetherian module.

If M is an R-graded module, we denote by $\text{K.dim}_R M$, the Krull dimension of M in the category R-mod, respectively in R-gr. (For details on the Krull dimension of a module see [6].)

It is well known that if M is a gr-noetherian module, then $\text{gr-K.dim}_R M$ exists.

Theorem 3.2. *Suppose that G is a strong polycyclic-by-finite group and M is an R-gr-noetherian module (hence an R-noetherian module). Then:*

$$\text{gr-K.dim}_R M \leq \text{K.dim}_R M \leq \text{gr-K.dim}_R M + h(G)$$

where $h(G)$ is the Hirsch number associated to G (see [13, p. 426]).

Proof. Let $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$ be a normal series for G. Taking into account the proof of Lemma 3 and the properties of the Krull dimension, it may be easily seen by induction on $0 \leq i \leq n$ that $\text{K.dim}_{R[G_{i+1}]} M[G_{i+1}] = \text{K.dim}_{R[G_i]} M[G_i]$ if G_{i+1}/G_i is a finite group and $\text{K.dim}_{R[G_{i+1}]} M[G_{i+1}] \leq \text{K.dim}_{R[G_i]} M[G_i] + 1$ if G_{i+1}/G_i is an infinite cyclic group. Now add these inequalities to obtain the required inequality.

4. A graded version of Maschke's theorem. Applications

In this paragraph (unless otherwise mentioned) $R = \bigoplus_{\sigma \in G} R_\sigma$ will be a graded ring of type G where G is a finite group with $n = \text{ord}\, G$.

Let $M = \bigoplus_{\sigma \in G} M_\sigma$, $N = \bigoplus_{\sigma \in G} N_\sigma$ be two $R[G]$-graded modules and $f \in \text{Hom}_{R\text{-gr}}(M, N)$. We define the map $\tilde{f}: M \to N$ by the equality:

$$\tilde{f}(x) = \sum_{g \in G} g^{-1} f(gx), \quad \forall x \in M.$$

Lemma 4.1. $\tilde{f} \in \text{Hom}_{R[G]\text{-gr}}(M, N)$.

Proof. It is easy to see that $\tilde{f}(M_\sigma) \subseteq N_\sigma$, $\forall \sigma \in G$.

Now we show that $\tilde{f}$ is an $R[G]$-homomorphism, i.e. $\tilde{f}(ax) = a\tilde{f}(x)$ for every $a \in R[G]$. It is clear that it is sufficient to prove that for $a = \lambda_\sigma \tau$, $\lambda_\sigma \in R_\sigma$, $\tau \in G$. Indeed,

$$\begin{aligned}\tilde{f}((\lambda_\sigma \tau)x) &= \sum_{g \in G} g^{-1} f((g\lambda_\sigma \tau)x) = \sum_{g \in G} g^{-1} f(\lambda_\sigma (g^{-1}\sigma g \tau)x) \\ &= \sum_{g} g^{-1} \lambda_\sigma f((\sigma^{-1} g \sigma \tau)x) = \sum_{g \in G} \lambda_\sigma \sigma^{-1} g^{-1} \sigma f((\sigma^{-1} g \sigma \tau)x).\end{aligned}$$

If we denote $h = \sigma^{-1} g \sigma \tau$, we have

$$\tilde{f}((\lambda_\sigma\tau)x)=\sum_{g\in G}(\lambda_\sigma\tau)(\sigma^{-1}g\sigma\tau)^{-1}f((\sigma^{-1}g\sigma\tau)x)$$
$$=\lambda_\sigma\tau\sum_{h\in G}h^{-1}f(hx)=(\lambda_\sigma\tau)\tilde{f}(x).$$

Proposition 4.2. *Let M be a graded $R[G]$-module and let $N\subseteq M$ be an $R[G]$-graded submodule of M. Assume that M has no n-torsion, where $n=\operatorname{ord} G$. If N is a direct summand of M in R-gr, then there exists an $R[G]$-graded submodule P of M such that $N\oplus P$ is essential in M as an R-module.*

Furthermore, if $M=nM$, then N is a graded direct summand of M as $R[G]$-module.

Proof (After the proof of [7, Lemma 1] or [8, Proposition 2.1]. We have $f\in\operatorname{Hom}_{R\text{-gr}}(M,N)$ such that $f(x)=x$, $\forall x\in N$. Let $\tilde{f}\in\operatorname{Hom}_{R[G]\text{-gr}}(M,N)$ as in Lemma 4.1. If $x\in N$, then $\tilde{f}(x)=nx$. We denote $P=\operatorname{Ker}\tilde{f}$; P is a graded $R[G]$-submodule of M. If $x\in P\cap N$, then $nx=0$ and by hypothesis we have $x=0$. Let $x\in M$; we denote $y=\tilde{f}(x)\in N$. Then $\tilde{f}(nx)=n\tilde{f}(x)=ny=\tilde{f}(y)$ and hence $\tilde{f}(nx-y)=0$ or $nx-y\in P$ and therefore $nx\in P\oplus N$ or $nM\subseteq N$. Hence $N\oplus P$ is essential in M as R-module. The second statement is clear.

If $M\in R$-gr, M is said to be gr-simple [9] if for every graded submodule N of M we have $N=0$ or $N=M$.

M is said to be gr-semi-simple if M is a direct sum of gr-simple modules. It is well known [9] that M is gr-semi-simple $\Leftrightarrow$ for any graded submodule N of M, N is a graded direct summand.

Theorem 4.3. *Let M be an R-gr-semi-simple module. If M has no n-torsion, then M is an R-semi-simple module.*

Proof. It is sufficient to prove the statement in the case M is gr-simple. We consider the graded $R[G]$-module $M[G]$. By the assertion (5) of Proposition 2.2, $M[G]$ is isomorphic to $\bigoplus_{\sigma\in G}M(\sigma^{-1})$ in the category R-gr. Hence $M[G]$ is an R-gr-semi-simple module. Since M is gr-simple and M has no n-torsion, $M=nM$, so $M[G]=nM[G]$.

By Proposition 4.2 it follows that $M[G]$ is an $R[G]$-gr-semi-simple-module.

By Theorem P, $(M[G])_e$ is an $(R[G])_e$-semi-simple module and hence M is an R-semi-simple module.

Corollary 4.4. *Let $R=\bigoplus_{\sigma\in G}R_\sigma$ be a graded ring of type G where $n=\operatorname{ord} G$ is invertible in R. If R is a gr-semi-simple artinian.*

For the R-graded module M we denote by $s(M)$, resp. $s_g(M)$, the socle of the R-module M, resp. the gr-socle of M (i.e. $s_g(M)=$ sum of gr-simple submodules of M).

Theorem 4.5. *Let $M=\bigoplus_{\sigma\in G} M_\sigma$ be an R-graded module where $n=\operatorname{ord} G$. Then:*

(1) $s(\underline{M})\subseteq s_g(M)$.

(2) $ns_g(M)\subseteq s(M)$.

(3) *If M has no n-torsion, then* $s_g(M)=s(M)$.

(4) *If* $x=\sum_{g\in G} x_g\in s(M)$ *where* $x_g\in M_g$, *then* $nx_g\in s(M)$, $\forall g\in G$.

Proof. (1) It is well known that $s(\underline{M})$ = intersection of all essential submodules of M. Analogously, we can show that $s_g(M)$ = intersection of all graded essential submodules of M. By Lemma I.2.8 of [9, p. 11], every graded essential submodule of M is an essential submodule. Hence $s(M)\subseteq s_g(M)$.

(2) Let $N\subseteq M$ be a gr-simple submodule of M. If $nN=0$, then $nN\subseteq s(M)$. If $nN\neq 0$, then because N is gr-simple, $N=nN$ and hence N has no n-torsion. By Theorem 4.3, it follows that N is a semisimple submodule of M and consequently $N\subseteq s(M)$. Hence $ns_g(M)\subseteq s(M)$.

(3) follows from (1) and (2).

(4) By assertion (1), we have that $x_\sigma\in s_g(M)$, $\forall\sigma\in G$. Now we apply statement (2) and we obtain that $nx_\sigma\in s(M)$ for any $\sigma\in G$.

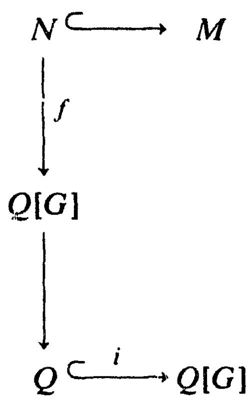
Corollary 4.6 [11, Theorem 2.2]. *Let $R=\bigoplus_{i\in\mathbb{Z}} R_i$ be a graded ring of type $\mathbb{Z}$ and let $M=\bigoplus_{i\in\mathbb{Z}} M_i$ be an R-graded module. Then the socle $s(\underline{M})$ is a graded submodule of M.*

Proof. Pick $x\in s(\underline{M})$ and decompose it as $x=x_{-k}+x_{-k+1}+\cdots+x_0+x_1+\cdots+x_l$, where $x_i\in M_i$. Let $n>l+k$; the $\mathbb{Z}$-graduation of R induces a $\mathbb{Z}/n\mathbb{Z}$-graduation in an obvious way. In this graduation the homogeneous components of x are exactly $x_{-k},x_{-k+1},\ldots,x_0,x_1,\ldots,x_l$. By Theorem 4.5 we have that $nx_i\in s(M)$.

Pick p,q two prime numbers such that they are both greater than $l+k$. Hence $px_i\in s(M)$ and $qx_i\in s(M)$. Since $(p,q)=1$, $x_i\in M$ and therefore $s(M)$ is a graded submodule of M.

Theorem 4.7. *Let $R=\bigoplus_{\sigma\in G} R_\sigma$ be a graded ring of type G. (G is a finite group). If $Q=\bigoplus_{\sigma\in G} Q_\sigma$ is a gr-injective object in the category R-*gr, *then $\underline{Q}$ is an injective module in R-*mod.

Proof. We show that $Q[G]$ is an injective object in the category $R[G]$-gr. Indeed, we consider the following diagram



where $M \in R[G]$-gr, N is an $R[G]$-graded submodule of M and $f \in \mathrm{Hom}_{R[G]\text{-gr}}(N, Q[G])$.

We define the canonical projection $\pi : Q[G] \to Q$ by $\pi(\sum_{g\in G} m_g g) = m_e$ where $m_g \in Q$ and $i : Q \to Q[G]$ by $i(m) = m \cdot e$.

If $x \in (Q[G])_\sigma$, then

$$x = \sum_{\lambda\mu=\sigma} m_\lambda \mu = \sum_{\mu\in G} m_{\sigma\mu^{-1}}\mu$$

and therefore $\pi(x) = m_\sigma$, so $\pi((M[G])_\sigma) \subseteq M_\sigma$. Hence $\pi \in \mathrm{Hom}_{R\text{-gr}}(Q[G], Q)$. It is clear that $i \in \mathrm{Hom}_{R\text{-gr}}(Q, Q[G])$. We denote $g = i \circ \pi \circ f \in \mathrm{Hom}_{R\text{-gr}}(N, Q[G])$. Since $Q[G] = \bigoplus_{\sigma\in G} Q(\sigma^{-1})$ in the category R-gr, $Q[G]$ is an injective object in the category R-gr. There exists an $h \in \mathrm{Hom}_{R\text{-gr}}(M, Q[G])$ such that $h(x) = g(x)$, $\forall x \in N$. We consider the morphism $\tilde{h} \in \mathrm{Hom}_{R[G]\text{-gr}}(M, Q[G])$ given by Lemma 4.1. Hence $\tilde{h}(x) = \sum_{\sigma\in G} \sigma^{-1} h(\sigma \cdot x)$, $x \in M$. If $x \in N_\lambda$, we have

$$\tilde{h}(x) = \sum_{\sigma\in G} \sigma^{-1} h(\sigma \cdot x) = \sum_{\sigma\in G} \sigma^{-1}(i \circ \pi \circ f)(\sigma \cdot x) = \sum_{\sigma\in G} \sigma^{-1} i(\pi(\sigma f(x))).$$

Since $f(x) \in (Q[G])_\lambda$, $f(x) = \sum_{\mu\in G} m_{\lambda\mu^{-1}}\mu$ and therefore

$$\sigma f(x) = \sum_{\sigma\in G} m_{\lambda\mu^{-1}} \mu\lambda^{-1}\sigma\lambda\mu^{-1}\mu = \sum_{\mu\in G} m_{\lambda\mu^{-1}}(\mu\lambda^{-1}\sigma\lambda).$$

If we denote $\mu\lambda^{-1}\sigma\lambda = \tau$, then $\sigma f(x) = \sum_{\tau\in G} m_{\sigma\lambda\tau^{-1}}\tau$ and therefore $\pi(\sigma f(x)) = m_\sigma\lambda$. On the other hand $\sigma^{-1} m_{\sigma\lambda} = m_{\sigma\lambda}(\lambda^{-1}\sigma^{-1}\sigma^{-1}\sigma\lambda) = m_\sigma\lambda(\lambda^{-1}\sigma^{-1}\lambda)$, so we have

$$\tilde{h}(x) = \sum_{\sigma\in G} \sigma^{-1} m_{\sigma\lambda} = \sum_{\sigma\in G} m_{\sigma\lambda}(\lambda^{-1}\sigma^{-1}\lambda) = \sum_{\mu\in G} m_{\lambda\mu^{-1}}\mu = f(x).$$

This means that $\tilde{h}(x) = f(x)$, $\forall x \in N$. Hence $Q[G]$ is an injective object in the category $R[G]$-gr. Now, by Theorem P, we obtain that $(Q[G])_e$ is an injective $(R[G])_e$-module and by Proposition 2.2 (4), Q is an injective module in R-mod.

Corollary 4.8. *Let $R = \bigoplus_{\sigma\in G} R_\sigma$ be a graded ring of type G (G is a finite group). If $M \in R$-gr, then*

$$\text{gr-inj.dim}_R M = \text{inj.dim}_R M.$$

Proof. If $0 \to M \to Q_0 \to Q_1 \to Q_2 \to \cdots$ is a minimal injective resolution of M in the category R-gr, then by Lemma I.2.8 of [9] and Theorem 4.7, this is a minimal injective resolution of $\underline{M}$ in R-mod.

The graded ring R is said to be gr-quasi Frobenius if ${}_R R$ is gr-artinian and gr-injective.

Corollary 4.9. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring of type G (G is a finite group). If R is gr-quasi Frobenius, then R is quasi-Frobenius.*

Proof. It is easy to see that ${}_R R$ is a left R_e-artinian module and consequently R is a left artinian ring. Now the statement follows immediately from Theorem 4.7.

Theorem 4.10. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring of type G where $n = \mathrm{ord}\, G$ is invertible in R. If R is a left gr-hereditary (resp. gr-semi-hereditary, resp. gr-regular Von Neumann) ring, then R is a left hereditary (resp. semi-hereditary, resp. regular Von Neumann) ring.*

Proof. Let K be a left graded ideal (resp. left finitely generated graded ideal) of R[G]. There exists a gr-free module L (resp. a gr-free module with finite basis) in the category $R[G]$-gr such that

$$L \xrightarrow{\varphi} K \to R[G]$$

where $\varphi \in \mathrm{Hom}_{R[G]\text{-gr}}(L, K)$ and φ is surjective.

Since L is a gr-free R-module (resp. L is a gr-free R-module with finite basis) and R is gr-hereditary (resp. gr-semi-hereditary), K is a gr-projective R-module and therefore there exists a $\sigma \in \mathrm{Hom}_{R\text{-gr}}(K, L)$ such that $\varphi \circ \psi = 1_K$.

Using Lemma 4.1, we consider the map $\tilde{\psi} \in \mathrm{Hom}_{R[G]\text{-gr}}(K, L)$. It is easy to see that $\varphi \circ (1/n)\tilde{\psi} = 1_K$ and hence K is a gr-projective $R[G]$-module. Consequently $R[G]$ is left gr-hereditary (resp. gr-semi-hereditary). Now we apply Theorem P to obtain the statement.

Analogously we show that gr-Von Neumann regular implies Von Neumann regular.

If $R = \bigoplus_{\sigma \in G} R_\sigma$ is a graded ring, we denote by gl.dim R (resp. gr-gl.dim R) the left homological global dimension of the category R-mod (resp. of the category R-gr).

Theorem 4.11. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring of type G where $n = \mathrm{ord}\, G$ is invertible in R. Then*

$$\text{gr-gl.dim}\, R = \text{gl.dim}\, R.$$

Proof. The inequality gr-gl.dim $R \leq$ gl.dim R is clear.

Suppose now that $t =$ gr-gl.dim R. Let $M \in R[G]$-gr and let

$$\cdots \to P_n \xrightarrow{f_n} \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

be a projective resolution of M in the category $R[G]$-gr.

Since $R[G]$ is a free R-module, the P_i are projective R-modules for every $i \geq 0$.

Since gr-gl.dim $R = t$, $K = \operatorname{Ker} f_{t-1} = \operatorname{Im} f_T$ is a projective R-module. Therefore there exists $g_t \in \operatorname{Hom}_{R\text{-gr}}(K, P_t)$ such that $f_t \circ g_t = 1_K$. Using Lemma 4.1, we consider the map $g_t \in \operatorname{Hom}_{R[G]\text{-gr}}(K, P_t)$. It is easy to see that $f_t \circ (1/n)\tilde{g}_t = 1_K$ and therefore K is a gr-projective $R[G]$-module. Hence gr-gl.dim $R[G] \leq t$. Now, by Theorem P, we obtain that gl.dim$(R[G])_e \leq t$ and consequently gl.dim $R \leq t$.

Remark. If n is not invertible, then Theorem 4.11 is false. (For example, if $R = R[G]$, where R is a field.)

If $M \in R$-gr and $K \subseteq M$ is a graded submodule of M, then K is called *gr-superfluous* (or *gr-small*) in M if the case for every graded submodule $L \subseteq M$, $K + L = M$ implies $L = M$.

Proposition 4.12. *Let $(M_i)_{i=1,\dots,n}$ be graded R-modules and $K_i \subseteq M_i$ be gr-superfluous modules in M_i, for every $i = 1, \dots, n$. Then $\bigoplus_{i=1}^n K_i$ is a gr-superfluous in $\bigoplus_{i=1}^n M_i$.*

Proof. Using the induction method, it is sufficient to prove the proposition for the case $n = 2$.

Let L be a graded submodule of $M_1 \oplus M_2$ such that $(K_1 \oplus K_2) + L = M_1 \oplus M_2$. It is easy to see that $K_1 + (K_2 + L) \cap M_1 = M_1$. Since K_1 is gr-superfluous in M_1, $(K_2 + L) \cap M_1 = M_1$, so $M_1 \subset K_2 + L$.

Since $K_1 \subset M_1 \subset K_2 + L \subset M_1 \oplus M_2$, $K_2 + L = M_1 \oplus M_2$. Hence $K_2 + (L \cap M_2) = M_2$. Since K_2 is gr-superfluous in M_2, $L \cap M_2 = M_2$, so $M_2 \subset L$ and therefore $M_1 \subset L$. Hence $L = M_1 \oplus M_2$.

Theorem 4.13. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring where G is a finite group and let M be a graded R-module and $K \subseteq M$ be a gr-superfluous submodule in M. Then K is a superfluous submodule in M.*

Proof. By Proposition 2.2(5) and Proposition 4.12, we have that $K[G]$ is a superfluous submodule in $M[G]$ as $R[G]$-module. Now, if we apply Theorem P, we obtain that $\psi(K) = (K[G])_e$ is a superfluous submodule in $\psi(M) = (M[G])_e$ as $(R[G])_e$-module. Hence K is a superfluous submodule in M.

Remark. Theorem 4.13 is false if the group G is infinite (see Remark 1.2.9 in [9, p. 10]).

If M is a graded module, we denote by $J_g(M)$ the intersection of all gr-maximal submodules of M; we call $J_g(M)$ the gr-Jacobson radical of M (see [9]).

We shall denote by $J(M)$ the Jacobson radical of M regarded without grading.

Corollary 4.14. *Let $R=\bigoplus_{\sigma\in G} R_\sigma$ be a graded ring where G is a finite group and let M be a graded finitely generated R-module. Then*

$$J_g(M)\subseteq J(M).$$

Proof. It is easy to see that $J_g(M)$ is the unique largest gr-superfluous submodule in M. By Theorem 4.13, the assertion follows.

Corollary 4.15. *Let $R=\bigoplus_{\sigma\in G} R_\sigma$ be a graded ring where G is a finite group and let M be a graded R-module. If $P\xrightarrow{f} M\to 0$ is a projective cover of M in the category R-*gr, *the $P\longrightarrow M\to 0$ is a projective cover of M in the category R-*mod.

Remarks 4.16. (1) All results in this section generalize their analogues given in [8] for strongly graded rings.

(2) It is easy to see that the converses of Theorems 4.3, 4.7, 4.10 and Corollaries 4.4 and 4.9 also hold.

(3) Let $R=\bigoplus_{i\in\mathbb{Z}} R_i$ be a graded ring of type $\mathbb{Z}$ such that it is left and right limited, i.e. there exist $m\in N$ with the property that $R_i=0$, $\forall i\in\mathbb{Z}$, $i\notin[-m,m]$.

For an arbitrary $n\in N$ such that $n>m$, the $\mathbb{Z}$-graduation of R induces a $\mathbb{Z}/n\mathbb{Z}$-graduation in an obvious way.

With new grading, the homogeneous components of R are the same with the ones in the initial grading.

This remark allows us to apply some of the results of this section to the case of $\mathbb{Z}$-graded rings which are left and right limited. Let us show, for example, that if $Q=\bigoplus_{i\in\mathbb{Z}} Q_i$ is a gr-injective R-module, then Q is an injective R-module. Indeed, let I be a left graded ideal of R (considered with the $(\mathbb{Z}/n\mathbb{Z})$-grading). Then I is a left graded ideal of R with the initial grading. Since Q is gr-injective, $\mathrm{Ext}^1_R(R/I,Q)=0$ and hence Q is gr-injective in the $(\mathbb{Z}/n\mathbb{Z})$-grading. Applying now Theorem 4.7, the assertion easily follows.

There are numerous examples of graded rings of type Z which are left and right limited. Here is one of them: let $(R, {}_RM_S, {}_SN_R, S)$ be a Morita context with maps $(\ ,\ ): M\otimes_S N\to R$ and $[\ ,\ ]: N\otimes_R M\to S$ (see for example [1, p. 62]). We consider the matrix ring

$$T=\begin{pmatrix} R & M\\ N & S\end{pmatrix}$$

in which the multiplication is defined by means of the mappings $(\ ,\ )$ and $[\ ,\ ]$. The ring T may be graded as follows:

$$T_0=\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}, \quad R_1=\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \text{ and } R_{-1}=\begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}.$$

With this grading, T is left and right limited.

5. The Jacobson radical

Lema 5.1. *$R=\bigoplus_{\sigma\in G} R_\sigma$, where G is a finite group.*

If $M=\bigoplus_{\sigma\in G} M_\sigma$ is a gr-simple object in R-gr, then M is semi-simple artinian of finite length in R_e-mod and $l_{R_e}(M)\leq n$, where $n=\operatorname{ord} G$.

Proof. If $M_\sigma\neq 0$, let $x\in M_\sigma$, $x\neq 0$. Since $Rx=M$, $Rx\cap M_\sigma=M_\sigma$. But $Rx\cap M_\sigma=R_e x$ and $R_e x=M_\sigma$. Hence M_σ is a simple R_e-module.

Lemma 5.2. *$R=\bigoplus_{\sigma\in G} R_\sigma$ is a graded ring, where G is a finite group. If $M\in R[G]$-gr is a simple object, then M is gr-semi-simple of finite length in the category R-gr.*

Proof. It is easy to see that ${}_R M$ is finitely generated. So there exists a gr-maximal submodule N of M in the category R-gr. Hence M/N is gr-simple in R-gr and by Lemma 5.1, M/N is a semi-simple R_e-module of finite length.

Since $M/N\simeq M/\sigma N$ in R_e-mod, we obtain that $M/\sigma N$ is a semi-simple R_e-module of finite length. (We remark that N is not an R-submodule of M.)

If we denote $N^*=\bigcap_{\sigma\in G}\sigma N$, then N^* is a graded $R[G]$-submodule of M. Indeed, if $a_g\in R_g$ and $h\in G$, then $a_g hN^*\subseteq a_g h\sigma N=g(h\sigma)g^{-1}a_g N\subseteq g(h\sigma)g^{-1}N$, so

$$a_g hN^*\subseteq \bigcap_{\sigma\in G} g(h\sigma)g^{-1}N=N^*.$$

Since M is a gr-simple module in $R[G]$-gr, $N^*=0$. Furthermore $0\to M\to \bigoplus_{\sigma\in G} M/\sigma N$ and we obtain that M is semi-simple of finite length in R_e-mod.

Now it is clear that M is gr-artinian and gr-noetherian in the category R-gr.

There exists an R-graded submodule $P\neq 0$ of M which is gr-simple. Now we define $f:P[G]\to M$, $f(x_g\sigma)=g\sigma g^{-1}\cdot x_g$ where $x_g\in P_g$, $\sigma\in G$. Clearly $f((P[G])_\sigma)\subseteq M_\lambda$, $\lambda\in G$. We show now that f is an $R[G]$-homomorphism. Indeed, if $a_\lambda\in R_\lambda$ and $\tau\in G$, we have

$$f((a_\lambda\tau)(x_g\sigma))=f(a_\lambda x_g g^{-1}\tau g\sigma)=g(g^{-1}\tau g\sigma)g^{-1}\lambda^{-1}(a_\lambda x_g)$$
$$=(\lambda\tau g\sigma g^{-1}\lambda^{-1})(a_\lambda x_g).$$

On the other hand

$$(a_\lambda\tau)\cdot f(x_g\sigma)=(a_\lambda\tau)(g\sigma g^{-1}x_g)=(a_\lambda g\sigma g^{-1})x_g$$
$$=(\lambda\tau g g^{-1}\lambda^{-1})(a_\lambda x_g)$$

and hence f is an $R[G]$-homomorphism. Now, since $M=\sum_{\sigma\in G}\sigma P$, it is easy to see that f is surjective.

By Proposition 2.2(4), $P[G]$ is gr-semi-simple in R-gr, so M is gr-semi-simple of finite length in R-gr.

Lemma 5.3. *Let $R=\bigoplus_{\sigma\in G}R_\sigma$ be a strongly graded ring of type G (G is a finite group). If $M=\bigoplus_{\sigma\in G}M_\sigma$ is a left graded R-module, then*

$$J_g(M)\cap M_e=J_{R_e}(M_e).$$

Proof. If $S=\bigoplus_{\sigma\in G}S_\sigma$ is a simple object in R-gr, then S is a semi-simple R_e-module (Lemma 5.1). Hence $J_{R_e}(M_e)\subseteq J_g(M)\cap M_e$. Conversely, let T be a simple R_e-module and $f:M_e\to T$ and R_e-homomorphism. By Theorem P, $S=R\otimes_{R_e}T$ is a simple object of R-gr. We have

$$M\simeq R\otimes_{R_e}M_e\xrightarrow{1\otimes f}R\otimes_{R_e}T$$

where $1\otimes f$ is an R-homomorphism. Hence

$$(1\otimes f)(J_g(M))=0$$

and because $J_g(M)=R\otimes_{R_e}(J_g(M)\cap M_e)$ we obtain that $R\otimes_{R_e}f(J_g(M)\cap M_e)=0$, so $R_e\otimes_{R_e}f(J_g(M)\cap M_e)=0$ and therefore $f(J_g(M)\cap M_e)=0$ or $J_g(M)\cap M_e\subseteq J_{R_e}(M_e)$.

Theorem 5.4. *Let $R=\bigoplus_{\sigma\in G}R_\sigma$ be a graded ring of type G where $n=\operatorname{ord}G<\infty$, and let $M=\bigoplus_{\sigma\in G}M_\sigma$ be a left graded R-module. Then:*

(1) $J_g(M)\subseteq J(M)$.

(1′) $J_g(M)=(J(M))_g$.

(2) $n\cdot J(M)\subseteq J_g(M)$.

(2′) *If $x=\sum_{\sigma\in G}x_\sigma\in J(M)$ where $x_\sigma\in M_\sigma$, then $nx_\sigma\in J(M)$ for any $\sigma\in G$.*

(3) *If n is invertible in R, then $J_g(M)=J(M)$.*

(4) $J(R)^nM\subseteq J_g(M)$.

(5) $J(R)\cap R_e=J_g(R)\cap R_e=J(R_e)$.

Proof. (1) Let U be a gr-simple object in the category $R[G]$-gr and $f\in\operatorname{Hom}_{R[G]\text{-gr}}(M[G],U)$. By Lemma 5.2, U is gr-semi-simple in R-gr. So $f(J_g(M))=(f\circ i)(J_g(M))=0$ where i is the inclusion morphism $i:M\to M[G]$. Hence $J_g(M)\subseteq J_g(M[G])$ and therefore $J_g(M)[G]\subseteq J_g(M[G])$.

By Proposition 2.2 and Lemma 5.3 we have

$$J_g(M)[G]\cap(M[G])_e\subseteq J_g(M[G])\cap(M[G])_e=J(M[G])_e$$

or $\psi(J_g(M))\subseteq\psi(J(M))$ and the assertion follows.

(1′) Let $x\in(J(M))_g$ be a homogeneous element and $f\in\operatorname{Hom}_{R\text{-gr}}(M,S)$ where S is a gr-simple R-module. Then $f(x)\in J(S)$, so $Rf(x)\subseteq J(S)$. If $f(x)\neq0$, then because $f(x)$ is a homogeneous element, $Rf(x)=S$ or $J(S)=S$, a contradiction (S is a finitely generated R-module).

(2) Let $x\in J(M)$ be an arbitrary element and $f\in \mathrm{Hom}_{R\text{-gr}}(M,S)$ where S is a gr-simple R module. If S has no n-torsion, then, by Theorem 4.3, S is a semi-simple R-module and therefore $f(x)=0$ or $f(nx)=0$.

If S has no n-torsion, then the homomorphism $\varphi_n : S\to S$, $\varphi_n(x)=nx$, has $\mathrm{Ker}\,\varphi_n\neq 0$. Since S is gr-simple, $\varphi_n=0$ or $nS=0$. Hence $f(nx)=0$ and consequently $nJ(M)\subseteq J_g(M)$.

(2′) follows directly from (2), and (3) follows from (1) and (2).

(4) If $S\in R$-gr is gr-simple, then, by Lemma 5.1 we obtain that M is artinian of finite length in R-mod. Moreover, $l_R(S)\leq l_{R_e}(S)\leq n$. Hence $J(R)^nS=0$ and thus clearly $J(R)^nM\subseteq J_g(M)$.

(5) It is clear that $J_g(R)\cap R_e=(J(R))_g\cap R_e$. Now, by statement (1′), we obtain the equality $J(R)\cap R_e=J_g(R)\cap R_e$. Let $I\subset R$ be a left gr-maximal ideal of R. Since $R_e/I_e\subset R/I$, we obtain, by Lemma 5.1, that R_e/I_e is a semi-simple artinian module in R_e-mod. Hence $J(R_e)\subseteq I_e$ and $J(R_e)\subseteq J_g(R)\cap R_e$.

If $a\in J_g(R)\cap R_e$ we have that $1-a$ is invertible in R and it is easy to see that $1-a$ is invertible in R_e. Hence $a\in J(R_e)$, so $J_g(R)\cap R\subseteq J(R_e)$.

Corollary 5.5. *If $R=\bigoplus_{i\in\mathbb{Z}}R_i$ is a graded ring of type $\mathbb{Z}$ and $M=\bigoplus_{i\in\mathbb{Z}}M_i$ a graded R-module, then $J(M)$ is a graded submodule of M.*

Proof. Let $x=\sum_{i\in\mathbb{Z}}x_i\in J(M)$ be an arbitrary element with $x_i\in M_i$. Let p,q be two prime numbers which are strictly greater than the number of non-zero homogeneous components of x. We consider R and M graded by the $(\mathbb{Z}/p\mathbb{Z})$-grading. By Theorem 5.4(2′) we have $px_i\in J(M)$, $\forall i\in\mathbb{Z}$.

Analogously, we obtain that $qx_i\in J(M)$. Since $(p,q)=1$, $x_i\in J(M)$.

Remarks 5.6. The assertions (1) and (2′) of Theorem 5.4 provide an answer to the question asked by G. Bergman in his paper [2]. We proved them in a more general case, namely for modules. The includion $J_g(R)\subseteq J(R)$ was proved for the first time by M. Cohen and S. Montgomery in [3].

The proof of Corollary 5.5 for $M=R$ was given by G. Bergman in [2] and the general case was presented in [12]. Another proof of the inclusion $J_g(M)\subseteq J(M)$ is contained in Corollary 4.14 using the study of gr-superfluous submodules (see Theorem 4.13).

6. Singular radical and prime radical

In this section $R=\bigoplus_{\sigma\in G}R_\sigma$ will be a graded ring of type G, where G is a finite group with $n=\mathrm{ord}\,G$. Using the graded version of Maschke's Theorem (see Section 4) we shall prove several properties of the singular and prime radical.

The main results in this paragraph are Corollary 6.3 and Theorem 6.5.

The assertion (1) and (4) of Theorem 6.5 were proved for the first time, using a different method, by M. Cohen and S. Montgomery in [3] (see Corollary 5.5 and 6.4 from [3]).

Lemma 6.1. *Let M be a graded $R[G]$-module and let $N \subseteq M$ be a graded R-submodule of M. We denote $N^* = \bigcap_{\sigma \in G} \sigma N$.*

(i) *N^* is a graded $R[G]$-submodule of M.*

(ii) *If N is an essential R-submodule of M, then N^* is essential in M as R-module.*

Proof. (i) is clear.

(ii) Assume that N is an essential R-submodule of M. Let $0 \neq x_g \in M_g$ be a nonzero homogeneous element of M and the set $G = \{\sigma_1, \ldots, \sigma_n\}$. Then $x = (\sigma_1 + \cdots + \sigma_n)x_g \in M$ and $x \neq 0$. By the proof of [9, Lemma I.2.8], there exists $a_\lambda \in R_\lambda$ such that $a_\lambda x \neq 0$ and $a_\lambda x \in N$. Hence $a_\lambda \sigma_i x_g \in N$, $\forall i = 1, 2, \ldots, n$. Since $a_\lambda \sigma = \lambda \sigma_i \lambda^{-1} a_\lambda$, $a_\lambda \sigma_i x_g = \lambda \sigma_i \lambda^{-1} a_\lambda x_g \in N$ or $a_\lambda x_g \in \lambda \sigma_i^{-1} \lambda^{-1} \cdot N$. Therefore $a_\lambda x_g \in \bigcap_{i=1}^n \lambda \sigma_i^{-1} \lambda^{-1} N = N^*$. Since $a_\lambda x \neq 0$, it is clear that $a_\lambda x_g \neq 0$.

Now, using again [9, Lemma I.2.8] it follows that N^* is an essential R-submodule of M.

Proposition 6.2. *Let $N \subseteq M$ be two graded $R[G]$-modules, having no n-torsion. Then*

(i) *There exists a graded $R[G]$-submodule $P \subseteq M$ with $N \oplus P$ essential in M as R-module.*

(ii) *N is essential in M as $R[G]$-module if and only if N is essential in M as R-module.*

Proof (After the proof of [8, Corollary 2.1] or [7, Lemma 2]).

(i) Let L be an R-graded submodule of M, maximal with respect to the property that $N \cap L = 0$. Then $N \oplus L$ is essential in M as R-module. Let $(N \oplus L)^* = \bigcap_{\sigma \in G} \sigma \cdot (N \oplus L)$. By Lemma 6.1, $(N \oplus L)^*$ is essential in M as R-module. If $K = (N \oplus L)^*$, then $N \subseteq K \subseteq N \oplus L$, so $K = N \oplus (K \cap L)$. By Proposition 4.2, there exists an $R[G]$-graded submodule P of K such that $N \oplus P$ is essential in K as R-module. Hence $N \oplus P$ is essential in M as R-module.

(ii) follows directly from (i).

Corollary 6.3. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring and $M \in R$-gr. Suppose that R has no n-torsion. Then $Z(M)$, the singular submodule of M, is a graded submodule of M.*

Proof. We consider the graded $R[G]$-module $M[G]$. By [8, Corollary 2.5] $Z_{R[G]}(M[G])$ is a graded submodule of $M[G]$ and

$$Z_{R[G]}(M[G]) \cap (M[G])_e = Z_{(R[G])_e}((M[G])_e).$$

Let $x \in Z_{R[G]}(M[G])$ be a homogeneous element of degree σ. Hence $x = \sum_{i=1}^{m} m_{g_i} h_i$ where $m_{g_i} \in M_{g_i}$, $h_i \in G$ and $g_i h_i = \sigma$ $(1 \le i \le m)$. Thus $I = 1_{R[G]}(x)$ is an essential left graded ideal of $R[G]$. By Proposition 6.2, I is an essential left ideal of $R[G]$ as an R-module, thus $J = I \cap R \subseteq R$ is an essential left ideal of R. But it is clear that $J \cdot m_{g_i} = 0$, so $m_{g_i} \in (Z(M))_g$, $\forall 1 \le i \le m$. Therefore $Z_{R[G]}(M[G]) \subseteq (Z(M))_g[G]$ and consequently

$$\psi(Z_R(M)) = Z_{(R[G])_e}(M[G])_e = Z(M)_g[G] \cap (M[G])_e = \psi((Z(M))_g)$$

and finally $z_R(M) \subseteq (Z(M))_g$ or $Z_R(M) = (Z(M))_g$.

Corollary 6.4. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring and $M \in R$-gr such that M has no n-torsion. If $N \subseteq M$ is an essential R-submodule of M, then $(N)_g$ is essential in M.*

Proof. Clearly $\psi(N)$ is essential in $\psi(M) = (M[G])_e$ as $(R[G])_e$-module. By Theorem P and [9, Lemma 1.2.8] we obtain that $R\psi(N)$ is essential in $M[G]$ as $R[G]$-module. Now, by Proposition 6.2(ii), $R\psi(N)$ is essential in $M[G]$ as R-module. Hence $M \cap R[G]\psi(N)$ is essential in M as R-module. If we apply Proposition 2.2(6), we have that$(N)_g$ is essential in M.

If R is a graded ring, a graded ideal I is graded prime if whenever $JK \subseteq I$ for J, K graded ideals of R, then $J \subseteq I$ or $K \subseteq I$.

The graded prime radical $\mathrm{rad}_g(R)$ is the intersection of all graded prime ideals of R. We denote by $\mathrm{rad}(R)$ the prime radical of R, i.e. the intersection of all prime ideals of R.

Proposition 6.5. *Let R be a graded ring and I a graded ideal of R.*

(i) *I is graded prime $\Leftrightarrow I = (P)_g$, the associated graded ideal of some prime P of R.*

(ii) $\mathrm{rad}_g(R) = (\mathrm{rad}(R))_g$.

(iii) *I is a graded prime ideal of $R \Leftrightarrow$ there exists a graded prime ideal J of $R[G]$ such that $I = J \cap R$.*

(iv) $\mathrm{rad}_g R = \mathrm{rad}_g(R[G]) \cap R$.

Proof. For the statements (i) and (ii), see [3, Lemma 5.1].

(iii) Let J be a graded prime ideal of $R[G]$ and I_1, I_2 two graded ideals of R such that $I_1 I_2 \subseteq J \cap R$. Then $I_1[G] \cdot I_2[G] \subseteq J$ and therefore $I_1[G] \subseteq J$ or $I_2[G] \subseteq J$, or $I_1 = I_1[G] \cap R \subseteq J \cap R$ or $I_2 = I_2[G] \cap R \subseteq J \cap R$. Hence $J \cap R$ is a graded prime ideal of R.

Conversely, let I be a graded prime ideal of R. Since $I[G] \cap R = I$, we can choose, by Zorn's Lemma, a graded ideal J of $R[G]$ maximal with respect to $J \cap R = I$. It is easy to see that J is a graded prime ideal of $R[G]$.

(iv) By statement (iii) we have that $\mathrm{rad}_g R \supseteq \mathrm{rad}_g(R[G]) \cap R$. Conversely, we prove that $(\mathrm{rad}_g R)[G] \subseteq \mathrm{rad}_g(R[G])$. For this, it is sufficient to show that if I is a graded ideal which is nilpotent, then $I[G]$ is a nilpotent ideal in $R[G]$. Here we show that if $I^m=0$, then $(I[G])^m=0$.

Indeed, it is sufficient to prove that if $a_{\sigma_1},\dots,a_{\sigma_m}\in I$ are homogeneous elements of I and $\tau_1,\dots,\tau_m\in G$, then $(a_{\sigma_1}\tau_1)(a_{\sigma_2}\tau_2)\cdots(a_{\sigma_n}\tau_m)=0$. Since $(a_{\sigma_1}\tau_1)(a_{\sigma_2}\tau_2)=a_{\sigma_1}a_{\sigma_2}(\sigma_2^{-1}\tau_1\sigma_2\tau_2)$, there exists an element $\tau\in G$ such that $(a_{\sigma_1}\tau_1)\cdots(a_{\sigma_m}\tau_m)=(a_{\sigma_1}\cdots a_{\sigma_m})\tau$ and thus $(a_{\sigma_1}\tau_1)\cdots(a_{\sigma_m}\tau_m)=0$.

Corollary 6.6. *Let $R=\bigoplus_{\sigma\in G}R_\sigma$ be a graded ring such that $n=\operatorname{ord}G$ is invertible in R. If $P\subsetneq I$ are two ideals of R, where P is a prime ideal, then $(P)_g\subsetneq(I)_g$.*

Proof. Clearly, $\varphi(P)\subsetneq\varphi(I)$ and $\varphi(P)$ is a prime ideal of $(R[G])_e$. Since any $g\in G$ commutes with all elements of $(R[G])_e$, $R[G]\cdot\varphi(P)$ is a graded prime ideal of $R[G]$. Analogously, $R[G]\varphi(I)$ is a two-sided ideal of $R[G]$ and $R[G]\varphi(P)\subsetneq R[G]\varphi(I)$. But it is easy to see that $R[G]\varphi(I)/R[G]\varphi(P)$ is essential in $R[G]/R[G]\varphi(P)$ as $R[G]$-module, and hence it is essential as R-module (Proposition 6.2). Using Proposition 2.2(6), we obtain that $(I)_g/(P)_g$ is essential in $R/(P)_g$ and therefore $(P)_g\subsetneq(I)_g$.

Theorem 6.7. *Let R be a graded ring of type G where $n=\operatorname{ord}G$.*

(1) *If R has no n-torsion, then* $\operatorname{rad}(R)=\operatorname{rad}_g(R)$.

(2) $n(\operatorname{rad}(R))\subseteq\operatorname{rad}_g(R)$.

(3) *If* $a=\sum_{g\in G}a_g\operatorname{rad}(R)$, $a_g\in R_g$, *then* $na_g\in\operatorname{rad}(R)$.

(4) $(\operatorname{rad}(R))^n\subseteq\operatorname{rad}_g(R)$.

Proof. (1) First, we shall prove the equality:

$$(\operatorname{rad}_g R)[G]=\operatorname{rad}_g R[G].$$

Replacing the ring R with the ring $R/\operatorname{rad}_g R$, it is sufficient to suppose that R is a graded semi-prime ring. (It is easy to see that the ring $R/\operatorname{rad}_g R$ has no n-torsion.) We shall prove then, that $\operatorname{rad}_g R[G]=0$. To continue the proof, we shall proceed as in [14, Theorem 2.2], using the graded version of Maschke's Theorem.

Let N be a nilpotent graded ideal of $R[G]$ and let $I=l_{R[G]}(N)$. Then I is a graded ideal of $R[G]$ and moreover, I is an essential right ideal in $R[G]$ (see the proof of [14, Theorem 2.2]). Now we apply Proposition 6.2 and conclude that $I\cap R$ is an essential right ideal in R, hence $I\cap R\neq 0$. Since $\operatorname{rad}_g(R)=0$, $l_R(I\cap R)=r_R(I\cap R)=0$. Since $I\cap R$ is a graded ideal of R, it is easy to see that $l_{R[G]}(I\cap R)=r_{R[G]}(I\cap R)=0$ and therefore $l_{R[G]}(I)=r_{R[G]}(I)=0$. Since $N\subseteq r_{R[G]}(I)$, $N=0$.

Consequently, we have $(\operatorname{rad}_g R)[G]=\operatorname{rad}_g R[G]$. By Proposition 2.1, $\varphi(\operatorname{rad}_g(R))=\operatorname{rad}_g R[G]\cap(R[G])_e$. By [8, Corollary 4.4], $\operatorname{rad}R[G]\cap R[G]_e=\operatorname{rad}(R[G])_e$. Since $\operatorname{rad}_g R[G]=(\operatorname{rad}R[G])_g$, it is clear that

$$\operatorname{rad}_g R[G]\cap(R[G])_e=\operatorname{rad}(R[G])_e=\varphi(\operatorname{rad}R).$$

So $\varphi(\operatorname{rad}_g(R))=\varphi(\operatorname{rad}R)$ and hence $\operatorname{rad}_g(R)=\operatorname{rad}(R)$.

(2) Let P be a graded prime ideal of R and $a\in\operatorname{rad}(R)$. If $n\cdot 1\in P$, then $na\in P$;

if $n\cdot 1\notin P$, then R/P has no n-torsion and thus, by statement (1) we have $\mathrm{rad}(R/P)=0$, so $\mathrm{rad}\, R\subseteq P$. Hence $na\in P$. Therefore $na\in \mathrm{rad}_g(R)$.

Assertion (3) follows from (1) and (2).

(4) The ring $R/\mathrm{rad}_g(R)$ is graded semi-prime. Exactly as in [7, Theorem 7] or [8, Theorem 4.2], we show that the ring $R[G]/\mathrm{rad}_g(R)[G]$ has a unique maximal nilpotent graded ideal N such that $N^n=0$. Hence $(\mathrm{rad}_g R[G])^n\subseteq \mathrm{rad}_g(R)[G]$ and applying again Proposition 2.1 we obtain that

$$\varphi((\mathrm{rad}\, R)^n)\subseteq(\mathrm{rad}_g(R)) \quad \text{or} \quad (\mathrm{rad}\, R)^n\subseteq \mathrm{rad}_g(R).$$

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